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Variations of a Charming Putnam Problem

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We will study the Lucas, Pell, and Pell–Lucas counterparts of a charming Fibonacci problem that appeared in the 68th annual William Putnam Mathematical Competition in 2007.

Introduction

It is well known that *Fibonacci numbers* and *Lucas numbers* satisfy the recurrence

$$x_n = x_{n-1} + x_{n-2},$$

where $n \geq 3$. When $x_1 = 1 = x_2$, $x_n = \mathrm{F}_n$, the nth Fibonacci number; and when $x_1 = 1$ and $x_2 = 3$, $x_n = \mathrm{L}_n$, the nth Lucas number. Clearly, we can extend the recurrence to include the case $n = 0$ and hence nonpositive subscripts. For example, $\mathrm{F}_0 = 0$ and $\mathrm{L}_0 = 2$. Table 1 shows the first ten Fibonacci and Lucas numbers.

Pell numbers P_n and *Pell–Lucas numbers* Q_n satisfy the recurrence

$$x_n = 2x_{n-1} + x_{n-2},$$

where $n \geq 3$. When $x_1 = 1$ and $x_2 = 2$, $x_n = \mathrm{P}_n$; and when $x_1 = 1$ and $x_2 = 3$, $x_n = \mathrm{Q}_n$. Pell and Pell–Lucas numbers can also be extended to nonpositive subscripts. Table 1 also shows the first ten Pell and Pell–Lucas numbers.

Binet-like formulas

Fibonacci and Lucas numbers can also be defined by the *Binet-like formulas* as follows:

$$\mathrm{F}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad \mathrm{L}_n = \alpha^n + \beta^n,$$

Table 1 The first ten Fibonacci, Lucas, Pell, and Pell–Lucas numbers.

n	F_n	L_n	P_n	Q_n
1	1	1	1	1
2	1	3	2	3
3	2	4	5	7
4	3	7	12	17
5	5	11	29	41
6	8	18	70	99
7	13	29	169	239
8	21	47	408	577
9	34	76	985	1 393
10	55	123	2 378	3 363

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

are solutions of the equation

$$t^2 - t - 1 = 0$$

(see reference 1). Both can be confirmed by solving the recurrence or using induction.

Pell and Pell–Lucas numbers have their own Binet-like formulas:

$$\mathrm{P}_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad \mathrm{Q}_n = \frac{\gamma^n + \delta^n}{2},$$

where

$$\gamma = 1 + \sqrt{2} \quad \text{and} \quad \delta = 1 - \sqrt{2}$$

are solutions of the equation

$$t^2 - 2t - 1 = 0$$

(see reference 2). Again, both follow by solving the recurrence or using induction.

A Putnam delight

The following charming problem appeared in the 68th annual William Putnam Mathematical Competition in 2007.

Problem 1 Find an explicit formula for x_n, where

$$x_{n+1} = 3x_n + \lfloor \sqrt{5} x_n \rfloor,$$

where $x_0 = 1$, $n \geq 0$, and $\lfloor x \rfloor$ denotes the *floor* of the real number x, the greatest integer $\leq x$.

The given recurrence, together with induction, can be used to show that

$$x_n = 2^{n-1} \mathrm{F}_{2n+3},$$

where $n \geq 0$ (see references 1 and 3). By virtue of the Binet formula for F_n, this gives an explicit formula for x_n. We encourage readers to construct a proof for themselves by adapting the proof of the Lucas counterpart given below.

The Lucas counterpart

We will now study the equally charming Lucas counterpart to problem 1.

Problem 2 Find an explicit formula for x_n, where

$$x_{n+1} = 3x_n + \lceil \sqrt{5} x_n \rceil,$$

where $x_0 = 2$, $n \geq 0$, and $\lceil x \rceil$ denotes the *ceiling* of the real number x, the smallest integer $\geq x$.

The first few values of x_n exhibit a clear pattern:

$$\begin{aligned} x_0 &= 2 = 2^{-1} \cdot 4, \\ x_1 &= 11 = 2^0 \cdot 11, \\ x_2 &= 58 = 2^1 \cdot 29, \\ x_3 &= 304 = 2^2 \cdot 76, \\ x_4 &= 1592 = 2^3 \cdot 199, \end{aligned}$$

and so on, where the last number on each line is a Lucas number.

Conjecture 1 *We have*

$$x_n = 2^{n-1} \mathrm{L}_{2n+3},$$

where $n \geq 0$*; this provides an explicit formula for* x_n*.*

Proof We will establish this using induction. The statement is clearly true when $n = 0$. Now assume that it holds for an arbitrary integer $n \geq 0$. Then we have

$$\begin{aligned} x_{n+1} &= 3x_n + \lceil \sqrt{5} x_n \rceil \\ &= \lceil (3 + \sqrt{5}) x_n \rceil \\ &= \lceil 2\alpha^2 x_n \rceil \\ &= \lceil 2\alpha^2 \cdot 2^{n-1} \mathrm{L}_{2n+3} \rceil \\ &= \lceil 2^n \alpha^2 (\alpha^{2n+3} + \beta^{2n+3}) \rceil \\ &= \lceil 2^n (\alpha^{2n+5} + \beta^{2n+1}) \rceil \\ &= \lceil 2^n [(\alpha^{2n+5} + \beta^{2n+5}) - (\beta^{2n+5} - \beta^{2n+1})] \rceil \\ &= 2^n \mathrm{L}_{2n+5} + \lceil -2^n \beta^{2n+3} (\beta^2 - \alpha^2) \rceil \\ &= 2^n \mathrm{L}_{2n+5} + \lceil 2^n \beta^{2n+3} (\alpha^2 - \beta^2) \rceil \\ &= 2^n \mathrm{L}_{2n+5} + \lceil 2^n \beta^{2n+3} \sqrt{5} \rceil. \end{aligned}$$

Since

$$2\beta^2 = 3 - \sqrt{5},$$

we have

$$2^n \beta^{2n} = (3 - \sqrt{5})^n.$$

So

$$2^n \beta^{2n+2} = \frac{(3 - \sqrt{5})^{n+1}}{2};$$

hence,

$$2^n \beta^{2n+3} = \frac{1 - \sqrt{5}}{2} \cdot \frac{(3 - \sqrt{5})^{n+1}}{2}.$$

Then

$$2^n\beta^{2n+3}\sqrt{5} = \frac{\sqrt{5}-5}{4}(3-\sqrt{5})^{n+1}.$$

This implies that

$$-1 < 2^n\beta^{2n+3}\sqrt{5} < 0.$$

Consequently,

$$\lceil 2^n\beta^{2n+3}\sqrt{5}\rceil = 0.$$

Thus,

$$x_{n+1} = 2^n \mathrm{L}_{2n+5} + 0 = 2^n \mathrm{L}_{2n+5};$$

so the formula works for $n+1$ also. Hence, it works for all $n \geq 0$, as desired.

Interestingly, the Fibonacci and Lucas versions have Pell and Pell–Lucas counterparts.

The Pell counterpart

We now present the Pell counterpart to problem 1.

Problem 3 Find an explicit formula for x_n, where

$$x_{n+1} = 3x_n + \lceil 2\sqrt{2}x_n\rceil,$$

$x_0 = 2$, and $n \geq 0$.

Conjecture 2 *Using the initial values* $x_0 = 2 = \mathrm{P}_2$, $x_1 = 12 = \mathrm{P}_4$, $x_2 = 70 = \mathrm{P}_6$, $x_3 = 408 = \mathrm{P}_8$, *and* $x_4 = 2\,378 = \mathrm{P}_{10}$, *we conjecture that* $x_n = \mathrm{P}_{2n+2}$, *where* $n \geq 0$.

Proof Clearly, the formula is true when $n = 0$. Now assume that it is true for an arbitrary integer $n \geq 0$. Then we have

$$\begin{aligned}
x_{n+1} &= 3x_n + \lceil 2\sqrt{2}x_n\rceil \\
&= \lceil (3+2\sqrt{2})x_n\rceil \\
&= \lceil \gamma^2 x_n\rceil \\
&= \lceil \gamma^2 \mathrm{P}_{2n+2}\rceil \\
&= \left\lceil \frac{\gamma^2}{\gamma-\delta}(\gamma^{2n+2}-\delta^{2n+2})\right\rceil \\
&= \left\lceil \frac{1}{\gamma-\delta}[(\gamma^{2n+4}-\delta^{2n+4}) + (\delta^{2n+4}-\delta^{2n})]\right\rceil \\
&= \mathrm{P}_{2n+4} + \left\lceil \frac{\delta^{2n+2}}{\gamma-\delta}(\delta^2-\gamma^2)\right\rceil \\
&= \mathrm{P}_{2n+4} + \lceil -2\delta^{2n+2}\rceil.
\end{aligned}$$

Since

$$\delta^2 = 3 - 2\sqrt{2},$$

we have

$$\delta^{2n} = (3 - 2\sqrt{2})^n;$$

so

$$-2\delta^{2n+2} = -2(3 - 2\sqrt{2})^{n+1};$$

hence, $-1 < -2\delta^{2n+2} < 0$. Consequently,

$$\lceil -2\delta^{2n+2} \rceil = 0.$$

This implies that

$$x_{n+1} = \mathrm{P}_{2n+4} + 0 = \mathrm{P}_{2n+4};$$

so the formula works for $n + 1$ as well. Thus, the formula works for all $n \geq 0$, as desired.

The Pell–Lucas counterpart

We now present the Pell–Lucas counterpart to problem 1.

Problem 4 Find an explicit formula for x_n, where

$$x_{n+1} = 3x_n + \lfloor 2\sqrt{2}x_n \rfloor,$$

$x_0 = 3$, and $n \geq 0$.

Using similar steps, we can show that $x_n = \mathrm{Q}_{2n+2}$, where $n \geq 0$. Again, we encourage readers to confirm the formula for x_n by adapting the proof for the Pell counterpart.

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