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# Making Averages Whole

JONNY GRIFFITHS

The arithmetic, geometric, and harmonic means of two natural numbers may or may not be integers. When are all three averages integers together?

What exactly do we mean by the average of two numbers? There are several candidates for this title. The other day, I was trying to explain on paper how the arithmetic, geometric, and harmonic means are defined. As a reminder, the AM of two positive numbers $x$ and $y$ is $(x+y)/2$, the GM is $\sqrt{xy}$, and the HM is $2xy/(x+y)$. It is worth noting that $\text{AM} \times \text{HM} = \text{GM}^2$, and also that $\text{AM} \geq \text{GM} \geq \text{HM}$, with equality holding if and only if $x = y$ (a visual demonstration of this can be found in reference 1). I picked a pair of values without much thought to make the AM a natural number, and then noticed that my choice made the GM a natural number too. The HM, it turned out, was *not* a whole number here, but the seed of a question had been planted: which pairs of natural numbers $(x, y)$ have an AM, a GM, and an HM that are all natural numbers?

It is not hard to find natural numbers that satisfy this condition; if $(x, y) = (45, 5)$, for example, then

$$\text{AM} = 25, \qquad \text{GM} = 15, \qquad \text{HM} = 9.$$

Clearly, if $x$ and $y$ are the same natural number, then the three averages will all be this natural number. We are looking for a parametrisation for $x$ and $y$ that gives all possible solutions, rather in the way that the parametrisation for Pythagorean triples (PTs) gives all possible primitive PTs. (If $x^2 + y^2 = z^2$, where $\gcd(x, y, z) = 1$ and where $x$ is even, then we have $x = 2mn$, $y = m^2 - n^2$, and $z = m^2 + n^2$, where $m$ and $n$ are natural numbers with $m > n$ of opposite parity, with no common factor. This parametrisation is derived in reference 2, p. 128.)

Let us start by saying $(x+y)/2 = a$, $\sqrt{xy} = b$, and $2xy/(x+y) = c$, where $a$, $b$, and $c$ are natural numbers, so $b^2 = ac$. We will also say that $x > y$ without loss of generality, since we have dealt with the case $x = y$. Then $x + y = 2a$ and $xy = b^2$, so $y = 2a - x$ and $x(2a - x) = b^2$. This yields

$$x^2 - 2ax + b^2 = 0.$$

Solving for $x$ we have $x = a + \sqrt{a^2 - b^2}$, $y = a - \sqrt{a^2 - b^2}$, and for these to be integers, $a^2 - b^2 = d^2$, for some natural number $d$. Substituting for $b^2$, we have $a^2 - ac - d^2 = 0$, which gives $a = (c + \sqrt{c^2 + 4d^2})/2$. Once again, for this to be an integer, $c^2 + 4d^2 = e^2$, for some natural number $e$. Let $\gcd(c, d) = f$. This yields $c = fp$, $d = fq$, $e = fr$, where $\gcd(p, q) = 1$, and so $p^2 + (2q)^2 = r^2$. Maybe now we can use the parametrisation of PTs outlined above.

There are two cases to consider. Firstly, if $p$ is odd (and so $r$ is odd), we can write $p = u^2 - v^2$, $q = uv$, $r = u^2 + v^2$ (where $u$ and $v$ are coprime and of opposite parity), and so

$$c = f(u^2 - v^2), \qquad d = fuv, \qquad e = f(u^2 + v^2).$$

Substituting back, $a = fu^2$, $b = fu\sqrt{(u^2 - v^2)}$, we obtain

$$x = fu^2 + fuv = fu(u + v), \qquad y = fu(u - v).$$

For $b$ to be an integer, $u^2 - v^2 = k^2$, for some integer $k$, or $u^2 = v^2 + k^2$ (note that $k$ must be odd, since $u$ and $v$ are of opposite parity). Once again using our PT parametrisation, we can put

$$u = m^2 + n^2, \qquad k = m^2 - n^2, \qquad v = 2mn,$$

for coprime integers $m, n$ that are of opposite parity. So we have the partial parametrisation

$$x = f(m^2 + n^2)(m + n)^2, \qquad y = f(m^2 + n^2)(m - n)^2.$$

Now for the second case, which is that $p$ is even, which means $q$ is odd (since $\gcd(p, q) = 1$), and $r$ is even. Let us put $p = 2s$, $r = 2t$, which gives that $s^2 + q^2 = t^2$. Since $q$ is odd, $s$ is even, since if $s$ and $p$ are both odd, $t^2 \equiv 2 \pmod 4$, which is impossible. We can now say

$$s = 2uv, \qquad q = u^2 - v^2, \qquad t = u^2 + v^2,$$

where $u$ and $v$ are coprime integers of opposite parity. This gives us $c = 2fs$, $d = fq$, $e = 2ft$, and so $c = 4uvf$ and $d = f(u^2 - v^2)$. So

$$e = 2f(u^2 + v^2), \qquad a = f(u + v)^2, \qquad b = \sqrt{a^2 - d^2} = 2f(u + v)\sqrt{uv},$$
$$x = a + d = 2uf(u + v), \qquad y = 2vf(u + v).$$

Thus, for $b$ to be an integer, we need $uv = k^2$. But we know that $\gcd(u, v) = 1$, and so for this to be true, we must have $u = m^2$, $v = n^2$, where $m$ and $n$ are coprime integers. Finally, we have the second half of the parametrisation, which is

$$x = 2f(m^2 + n^2)m^2, \qquad y = 2f(m^2 + n^2)n^2.$$

We have shown that any parametrisation that meets our conditions must be of these forms; it remains to check that our two formulations do in fact give three integer averages, and they do.

There are some maths problems that seem to set out to wilfully make life difficult for you, while others seem to fall over themselves to be obliging. I would say that this question is one of the second type – the way that repeatedly using the PT formula proves so profitable, and the way the facts concerning coprimeness work so neatly to our advantage, is pleasing. How many of these $(x, y)$ pairs are there? A computer search reveals that there are 58 $(x, y)$ pairs with $1000 \geq x > y$.

**References**


1 J. Griffiths, Carom 1-2: Inequalities, available at http://www.s253053503.websitehome.co.uk/carom/carom-files/carom-1-2.ppt.

2 J. Griffiths, Lyness cycles, elliptic curves, and Hikorski triples, MSc Thesis, University of East Anglia, 2012, available at http://www.s253053503.websitehome.co.uk/jg-msc-uea/thesis-final-11-2-2012.pdf.



***Jonny Griffiths*** *taught mathematics at Paston Sixth Form College in Norfolk for over twenty years. He has studied mathematics and education at Cambridge University, the Open University, and the University of East Anglia. Possible claims to fame include being a member of 'Harvey and the Wallbangers', a popular band in the 1980s, and playing the character Stringfellow on the childrens' television programme 'Playdays'. He currently works for the Cambridge Mathematics Education Project.*