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Twelve New Points on the Nine-Point Circle

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The authors take another look at a famous circle.

Let $\triangle ABC$ be a given triangle, D, E, F be the midpoints of the sides BC, AC, AB, respectively. Let AG, BH, CI be the altitudes on the sides BC, AC, AB and G, H, I be the perpendicular feet. If O is the orthocentre and J, K, L are the midpoints of the segments AO, BO, CO, then the nine points D, E, F, G, H, I, J, K, L are on a circle.

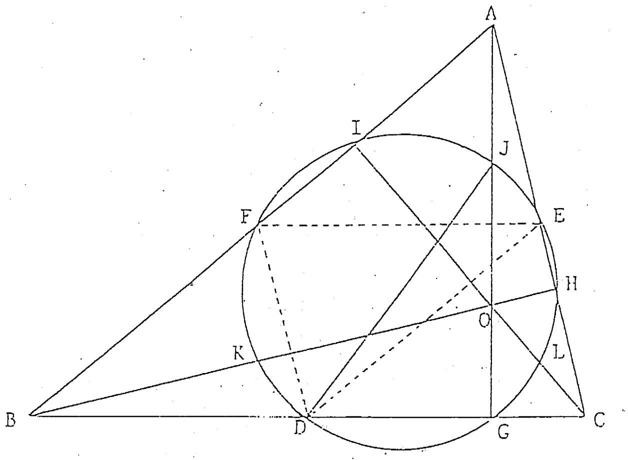

Figure 1

This nine-point circle theorem was discovered in the 19th century. The name was given by Poncelet (reference 1), and is commonly used in English-speaking countries. In France it is usually called Euler's circle, while in Germany it is called Feuerbach's circle (references 1 and 2).

In reference 3, three more points are added to the nine-point circle. They are the intersection points of the perpendicular bisectors of BC, AC, AB and the perpendicular bisectors of AO, BO, CO, respectively. Therefore, there are so far twelve special points on the nine-point circle.

It is easily seen that the altitudes of the triangle play a decisive role in the nine-point circle theorem. Six of the nine points are on the altitudes. For a triangle, medians and angle bisectors are just as important as altitudes. It is a little unreasonable that they have no contribution to this fantastic circle. In this note we prove that there are twelve new points on the nine-point circle by the help of medians and angle bisectors.

Before stating the result, we first point out a fact we are going to use. Referring to figure 1, DJ is a diameter of the nine-point circle because $\angle DGJ$ is a right angle. We assume that the radius of the nine-point circle is 1 and that $\angle C > \angle B$. Then

$$\angle DJG = \tfrac{1}{2}\text{arc}\,DG = \tfrac{1}{2}(\text{arc}\,DE - \text{arc}\,GE).$$

Also

$$\angle DFE = \tfrac{1}{2}\text{arc}\,DE \quad \text{and} \quad \angle EDC = \tfrac{1}{2}\text{arc}\,GE,$$

so that

$$\angle DJG = \angle DFE - \angle EDC.$$

However, since the triangles DEF, EDC, FBD are similar,

$$\angle DFE = \angle C \quad \text{and} \quad \angle EDC = \angle B.$$

Hence, finally,

$$\angle DJG = \angle C - \angle B.$$

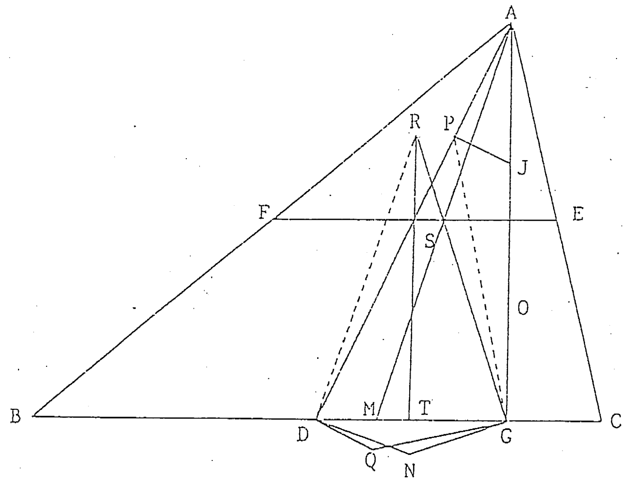

Figure 2

Theorem. Let $\triangle ABC$ be a triangle, D, E, F be the midpoints of the sides BC, AC, AB. Let AD, AG, be the median and the altitude on the side BC, and AM be the angle bisector of $\angle A$. If the orthocentre of $\triangle ABC$ is O and J is the midpoint of the segment AO, then the following statements are true.

(1) The intersection point P of the median AD with its perpendicular line passing through J is on the nine-point circle.

(2) The intersection point Q of the line passing through D and perpendicular to AD, and the line passing through G and perpendicular to PG, is on the nine-point circle.

(3) Let S be the intersection point of EF with the angle bisector AM. Then the intersection point R of the line GS with the perpendicular bisector of DG is on the nine-point circle.

(4) The intersection point N of the line passing through D and perpendicular to DR, and the line passing through G and perpendicular to GR, is on the nine-point circle.

Proof. (1) In figure 2, since $JP \perp AD$ and $AG \perp DG$, the four points P, J, G, D must be on a circle. Hence P is on the circle passing through J, G, D. This circle is the nine-point circle.

(2) Since $QP \perp PD$ and $QG \perp PG$, the four points P, D, Q, G must be on a circle. Hence Q is on the circle passing through P, D, G. By (1), we know that this circle is the nine-point circle.

(3) Let T be the midpoint of the segment DG. Since S is on the segment EF, which is the perpendicular bisector of AG, $AS = GS$ and $\angle SAG = \angle SGA$. Since AG is the altitude on BC and RT is the perpendicular bisector of DG, we have that TR is parallel to AG and $\angle TRG = \angle SGA$. Since R is on the perpendicular bisector of DG, we have $RG = RD$ and $\angle TRG = \angle TRD$. But $\angle SAG = \angle A/2 - \angle CAG = \angle A/2 - (\pi/2 - \angle C) = (\angle C - \angle B)/2$. Therefore $\angle DRG = \angle C - \angle B$. By the statement above the theorem, $\angle DJG = \angle C - \angle B$. Thus, $\angle DRG = \angle DGJ$ and the four points R, J, G, D are on a circle. The point R is thus on the circle passing through J, G, D. This circle is the nine-point circle.

(4) Since $DN \perp DR$ and $NG \perp GR$, the four points R, D, N, G are on a circle. Thus N is on the circle passing through R, D, G. By (3), this circle is the nine-point circle.

Remark. If we discuss the sides AC, AB we can get eight more new points on the nine-point circle. Therefore we have twelve new points on the nine-point circle. Together with the twelve points already obtained, there are twenty-four special points constructed easily from $\triangle ABC$ by the help of altitudes, medians and angle bisectors. There could be more special points discovered in the future. Therefore, the name of the nine-point circle does not reflect the fact very well. It is high time to rename it as either Euler's circle, for short, or the Euler–Poncelet–Feuerbach circle to reflect the history of discovery.

References

1. N. Altshiller-Court, *College Geometry*, (Barnes and Noble, New York, 1952).
2. H. W. Eves, *Fundamentals of Geometry*, (Allyn and Bacon, Boston, 1969).
3. J. Tong and S. Kung, Proof without words: the nine-point circle is in fact a twelve-point circle, submitted. □

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