The Loneliness of the Factorions

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Number is the bond of the eternal continuance of things. Plato

Factorions are numbers that are the sum of the factorial values for each of their digits. For example, 145 is a factorion because it can be expressed as

$$145 = 1!+4!+5!.$$

Two tiny factorions are

$$1 = 1! \quad \text{and} \quad 2 = 2!.$$

The largest known factorion is 40 585; it can be written as

$$40\,585 = 4!+0!+5!+8!+5!.$$

(40 585 was discovered in 1964 by R. Dougherty using a computer search.)

Can you end the loneliness of the factorions? Do any others exist?

In fact these four factorions are the *only* ones known.

Various proofs have been given to show that 40 585 is the largest possible factorion. To indicate why this is true we need only realize that if

$$A = a_r a_{r-1} \dots a_1 a_0$$
$$= a_r \times 10^r + \cdots + a_1 \times 10 + a_0$$

is a factorion, then

$$10^r \leqslant a_r \times 10^r \leqslant A = a_r! + \cdots + a_1! + a_0!$$
$$\leqslant 9!\,(r+1) = 362\,880(r+1).$$

Thus $r \leqslant 6$. In fact, if $r = 6$ then we easily find that $a_6 = 1$ and $A \leqslant 1999\,999$. (A little more work shows that $A \leqslant 999\,999$.) It is then straightforward to check on a computer that 40 585 is the last factorion.

A more fruitful avenue of research may be the search for factorions 'of the second kind' which are formed by the *product* of the factorial values for each of their digits. Additionally, there are hypothetical factorions 'of the third kind' formed by grouping digits. For example, a factorion of the third kind might have the form

$$abcdef = (ab)! + c! + d! + (ef)!,$$

where each letter represents a digit.

To date, I am unaware of the existence of factorions of the second or third kind, and I would be interested in hearing from readers who can find any. (I should point out Herve Bronninan from Princeton University has recently found some magnificent factorions in other bases, most notably 519 326 767, which in base 13 is written as 8.3.7.9.0.12.5.11 and is equal to $8!+3!+7!+9!+0!+12!+5!+11!$. You can interpret this base 13 number as:

$$8\times13^7+3\times13^6+7\times13^5+9\times13^4$$
$$+0\times13^3+12\times13^2+5\times13^1+11\times13^0.)$$

Digressions: Narcissistic numbers

Dik T. Winter from Amsterdam is an expert on a somewhat related class of numbers which are the sums of powers of their digits. In other words, these are N-digit numbers which are equal to the sum of the Nth powers of their digits. For example,

$$153 = 1^3+5^3+3^3.$$

Variously called narcissistic numbers, 'numbers in love with themselves', Armstrong numbers or perfect digital variants, these kinds of numbers have fascinated number theorists for decades. For example, the English mathematician G. H. Hardy (1877–1947) noted that 'There are just four numbers, after unity, which are the sums of the cubes of their digits .... These are odd facts, very suitable for puzzle columns and likely to amuse amateurs, but there is nothing in them which appeals to the mathematician.' I gave 153 as an example of such a number. Can you find the other three?

The largest narcissistic number discovered to date is the incredible 39-digit number:

$$115\,132\,219\,018\,763\,992\,565\,095\,597\,973\,971\,522\,401.$$

(Each digit is raised to the 39th power.) Can you beat the world record? What would G. H. Hardy have thought of this multidigit monstrosity?

The frequency of occurrence of narcissistic numbers varies according to the base of the number system in which one conducts a search. For example, in our standard (base 10) system there are 88 known numbers of this type, while in base 4 there are only 11. Table 1 shows some numbers from Dik T. Winter. The number of digits for the largest number known is in parentheses.

As one searches for larger and larger narcissistic numbers, will they eventually run out, as in the case of the lonely factorials?

Finally, Kevin S. Brown writes that he knows of only three occurrences of $n!+1 = m^2$, namely

$$25 = 4!+1 = 5^2,$$

$$121 = 5!+1 = 11^2,$$

$$5041 = 7!+1 = 71^2.$$

We do not know if there are any others. Perhaps these 'Brown numbers' will be as lonely as the factorions. Prolific mathematician Paul Erdős long ago conjectured that there are only three such numbers. Erdős offers a cash prize for a proof of this—see *Mathematical Spectrum* Volume 27 Number 2 pages 43–44.

Reference

C. Pickover, *Keys to Infinity* (Wiley, New York, 1995). □

Table 1

Base	Total	Digits	Largest
2	1	(1)	1
3	5	(3)	122
4	11	(4)	3 303
5	17	(14)	14 421 440 424 444
6	30	(18)	105 144 341 423 554 535
7	59	(23)	12 616 604 301 406 016 036 306
8	62	(29)	11 254 613 377 540 170 731 271 074 472
9	58	(30)	104 836 124 432 728 001 478 001 038 311
10	88	(39)	115 132 219 018 763 992 565 095 597 973 971 522 401
11	134	(45)	123 44A A12 A72 180 342 291 2A8 AA4 963 568 083 A26 845 6A4
12	87	(51)	150 793 46A 6B3 B14 BB5 6B3 958 98B 966 29A 8B0 151 534 4B4 B07 14B

($A = 10$, $B = 11$.)

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