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The Undulation of the Monks

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Such as say that things infinite are past God's knowledge may just as well leap headlong into this pit of impiety, and say that God knows not all numbers .... What madman would say so? .... What are we mean wretches that dare presume to limit His knowledge.

St Augustine

The number 69 696 is a remarkable number and certainly my favourite of all the integers. Aside from being almost exactly equal to the average velocity in miles per hour of the Earth in orbit, it is also the surface temperature, in degrees Fahrenheit, of some of the hottest stars. More important are its fascinating mathematical properties.

It is written that a Tibetan monk once presented this number to a student and said: 'What do you find significant about 69 696?'

The student thought for a few seconds, and replied, 'That is too easy, Master. It is the largest undulating square known to humanity.'

The teacher pondered this answer, and himself started to undulate in a mixture of excitement and perhaps even terror.

To understand the monk's passionate response, we must digress to some simple mathematics. *Undulating numbers* are of the form

$$ababababab\ldots.$$

For example, 171 717 and 28 282 are undulating numbers. When I first conceived the idea of undulating squares a few years ago, it was not known if any such numbers existed. It turns out that $69\,696 = 264^2$ is indeed the largest undulating square known to humanity, and most mathematicians believe we will never find a larger one.

Dr Noam D. Elkies from the Harvard Mathematics Department wrote to me about the probabilities of finding undulating squares. The chance that a 'random' number around x is a perfect square is about $1/\sqrt{x}$. More generally, the probability is $x^{-1+1/d}$ for a perfect dth power. Since there are (for any k) only 81 k-digit undulants, one would expect to find very few undulants that are also perfect powers, and none that are very large. Dr Elkies believes that listing all cases may be impossible using present-day methods for treating exponential Diophantine equations.

Bob Murphy used the software Maple V to search for undulating squares, and he discovered some computational tricks for speeding the search. For example, he began by examining the last 4 digits of perfect squares (i.e. he computed squares modulo 10 000). Interestingly, he found that the only possible digit endings for squares which undulate are: 0404, 1616, 2121, 2929, 3636, 6161, 6464, 6969, 8484 and 9696. By examining squares modulo 100 000, then modulo 1 000 000, then modulo 10 000 000, etc., he found that no perfect square ends in 40 404, 6 161 616, 63 636, 464 646 464 or 969 696, thereby allowing him to speed further the search process. Searching all possible endings, he asserts that, if there is an undulating square, it must have more than 1000 digits.

Dr Helmut Richter from Germany is the world's most famous undulation hunter, and he has indicated to me that it is not necessary to restrict the 'modular searches' to powers of 10, and arbitrary primes work very well. He has searched for undulating squares with a million digits or fewer, using a Control Data Cyber 2000. No undulating squares greater than 69 696 have been found.

Other undulants

Randy Tobias of the SAS Institute in North Carolina notes that there are larger undulating squares in other number bases. For example, $292^2 = 85\,264 = 41\,414$ in base 12. And 121 is an undulating square in any base. (121 in base n is $(n+1)^2$.)

Interestingly, we find that there are very few undulating powers of any kind in base 10. For example, a 3-digit undulating cube is $7^3 = 343$. However, Randy Tobias conducted a search for other undulating powers and found that 343 is the *only* undulant he could find that is formed by raising a number to a power $p \geqslant 3$. He has checked this for $3 \leqslant p \leqslant 31$ and for for undulants less than 10^{100}. Undulating powers are indeed rare!

Undulating prime numbers, on the other hand, are more common. For example, Randy has discovered the following huge undulating prime:

$$7+720\times\frac{100^{49}-1}{99}$$

$$= 727\,272\,727\,272\,727\,272\,727\,272\,727\,272\,727$$
$$727\,272\,727\,272\,727\,272\,727\,272\,727\,272\,727$$
$$727\,272\,727\,272\,727\,272\,727\,272\,727\,272\,727$$

(It has 99 digits.) To find this monstrosity, he also used the software program called Maple. The program scanned numbers using two lines:

$$(a*10+b)*(10**(2*(k+1))-1)/99$$
$$a+10*(a*10+b)*(10**(2*(k+1))-1)/99$$

for $0 \leqslant k \leqslant 50$, $1 \leqslant a \leqslant n-1$ and $0 \leqslant b \leqslant n-1$. The Maple **isprime()** function was used to check if a number is prime. Maple makes it possible to work with very large integers.

There are many other undulating primes with many digits. However, there is no undulating prime with an even number of digits, because $abab\,ab\ldots ab = ab\times 101\,01\ldots 01$. I would be interested in hearing from readers who have searched for undulating primes with larger periods of undulation, such as found in the prime number 59 959 959 959 (which does not finish its last cycle of undulation).

Binary undulants beyond imagination

Finally, *binary undulants* are powers of 2 that alternate the adjacent digits 1 and 0 somewhere in their decimal expansion. For example, the 'highest quality' binary undulant I have found is 2^{949}. It has the undulating binary sequence 101 010 in it, which I have placed in parentheses in the following:

$$2^{949} = 4758\,454\,107\,128\,905\,800\,953\,799\,994\,079$$
$$681\,792\,420\,032\,645\,310\,062\,268\,978\,469$$
$$949\,81(1\,010\,10)2\,913\,293\,995\,344\,538\,606$$
$$387\,700\,321\,887\,355\,916\,128\,617\,513\,761$$
$$454\,672\,785\,743\,698\,264\,930\,657\,859\,527$$
$$662\,802\,505\,506\,689\,431\,871\,596\,616\,596$$
$$511\,469\,752\,757\,984\,765\,426\,503\,524\,599$$
$$059\,416\,795\,862\,009\,216\,282\,102\,716$$
$$609\,115\,705\,865\,638\,544\,337\,453\,260$$
$$521\,036\,049\,116\,206\,989\,312.$$

Here 949 is called an undulation seed of order 6, since it gives rise to a 6-digit undulation pattern of adjacent 1's and 0's. When I challenged mathematicians and programmers around the world to produce a higher-order binary undulant, many took up the challenge. The highest quality binary undulant so far known to humanity was discovered by Arlin Anderson of Alabama. He was the first to find that 2^{1802} contains an 8-digit binary undulation. After much hard work he also found that $2^{7694\,891}$ starts with the digits 101 010 101 73..., and a week later he discovered that $2^{1748\,219}$ gives rise to a 10-digit undulant! Since Arlin only checked the last 240 digits of each number, he feels it is almost certain that there is a bigger binary undulation somewhere in the first million powers of 2. Considering that $2^{1000\,000}$ contains around 300 000 digits, the chance of finding a 10 101 010 101 or 01 010 101 010 is large. (Arlin uses a custom C-program for large integer computation. The program runs on an Intergraph 6040 Unix workstation and on a 486 PC. Searching 240 digits in 2 million powers of 2 required 15 hours.)

How do binary undulants vary with the base b? For example, for the case of $b = 2$, all powers of 2 are binary undulants.

Further Reading

1. C. Pickover, Is there a double smoothly undulating integer? In *Computers, Pattern, Chaos and Beauty* (St. Martin's Press, New York, 1990).
2. C. Pickover, *Keys to Infinity* (Wiley, New York). □

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