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When the Coefficients are the Roots

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When are the coefficients of a polynomial equation equal to its roots?

The other day I ran into this question:

> if the roots of the quadratic equation $x^2 + ax + b = 0$ are a and b, what are the possible values for a and b?

The question is straightforwardly solved. We say $x^2 + ax + b = (x - a)(x - b) = x^2 - (a + b)x + ab$. This is true for all values of x, so we can equate the coefficients of each side, giving $-a - b = a$, and $ab = b$. Thus, on solving, the possibilities for (a, b) are $(0, 0)$ or $(1, -2)$.

So far, so good, but the question to my mind immediately invited generalisation. If we are given the equation $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 = 0$, then what possibilities for $(a_{n-1}, a_{n-2}, \ldots, a_0)$ are there if these values are also the roots of the equation?

We can note firstly that $(0, 0, \ldots, 0)$ will always be a solution (we will call this the *trivial solution*).

Secondly, if $(a_{n-1}, a_{n-2}, \ldots, a_0)$ is a solution for the degree-n equation, then by multiplying our degree-n equation by x, we see that $(a_{n-1}, a_{n-2}, \ldots, a_0, 0)$ will be a solution in the degree-$(n + 1)$ case. Thus, if we add 0s as we go, the number of solutions we find will be nondecreasing as n increases.

What happens with the degree-3 case, the cubic equation? We have $x^3 + ax^2 + bx + c = (x - a)(x - b)(x - c)$, and on expanding and comparing coefficients, we find

$$[2a + b + c, ab + ac + bc - b, c(ab + 1)] = [0, 0, 0].$$

Let me explain the notation here. The algebra rapidly becomes unfriendly with this problem, and the use of a computer algebra package becomes essential. I am using Derive 6, but there are many alternatives – if you do not currently use such a program, it will open up a wealth of mathematics investigations to you if you do. When it comes to algebra, Derive is infinitely quicker and more accurate than I can ever hope to be.

Here, $[x, y, z]$ denotes a three-dimensional vector, a format which makes the substitution of values within Derive easy. We effectively have three equations in three unknowns. Solving $2a + b + c = 0$ gives $c = -2a - b$ and, substituting in, we have $[0, -2a^2 - 2ab - b^2 - b, -(2a + b)(ab + 1)] = [0, 0, 0]$. Now, putting $(2a + b)(ab + 1) = 0$, we have $b = -2a$ or $b = -1/a$.

Taking the first of these options gives $[0, 2a(1-a), 0] = [0, 0, 0]$, and so $a = 0$ or 1. These give us the trivial solution $(0, 0, 0)$, and $(1, -2, 0)$. Taking $b = -1/a$ instead gives $[0, -2a^2 + 1/a - 1/a^2 + 2, 0]$, and solving for a here gives four solutions, two complex (we will ignore these), $a = 1$ and (curiously) $a = 0.565\,197\,717\,3\ldots$. Substituting back, we get the triplets for (a, b, c) of $(1, -1, -1)$ and $(0.565\,197\,717\,3\ldots, -1.769\,292\,354\ldots, 0.638\,896\,919\,3\ldots)$.