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Four Functions and a Bijection

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By applying the floor and ceiling operations to real numbers, a real function can be discretized to give a new function on the integers. Much information will, of course, be lost in such a process, but perhaps some intrinsic property concerning order may be retained. Even for the integers, inequalities can be tricky, and the article may serve as a useful exercise in their manipulation.

1. Four integer functions from a real bijection

Let $\mathbb{U} = [1, \infty)$, the set of real numbers, $x \geq 1$, and $\phi\colon \mathbb{U} \to \mathbb{U}$ be an increasing bijection. The inverse ψ of ϕ is also such a bijection, and the two composite functions $\psi \circ \phi$ and $\phi \circ \psi$ are the identity function on $\mathbb{U}$, that is $\psi(\phi(x)) = x$ and $\phi(\psi(t)) = t$, for $x, t \in \mathbb{U}$.

Let $\mathbb{N} = \{1, 2, \ldots\}$, the set of positive integers. For $x \in \mathbb{U}$, the floor $\lfloor x \rfloor$ is the greatest integer $n \leq x$, and the ceiling $\lceil x \rceil$ is the least integer $n \geq x$, so that $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$ with $\lfloor x \rfloor, \lceil x \rceil \in \mathbb{N}$; thus, $\lceil x \rceil = \lfloor x \rfloor + 1$, unless $x \in \mathbb{N}$, in which case $\lfloor x \rfloor = \lceil x \rceil$. Next, for a function $f\colon \mathbb{U} \to \mathbb{U}$, we define $f^+, f^-\colon \mathbb{N} \to \mathbb{N}$ by

$$f^+(n) = \lceil f(n) \rceil, \quad f^-(n) = \lfloor f(n) \rfloor, \qquad n \in \mathbb{N}.$$

Returning to ϕ and ψ, there are now four composite functions $f_i\colon \mathbb{N} \to \mathbb{N}$, $i = 1, 2, 3, 4$, given by

$$f_1(n) = \psi^-(\phi^+(n)), \quad f_2(n) = \psi^+(\phi^-(n)),$$
$$f_3(n) = \phi^-(\psi^+(n)), \quad f_4(n) = \phi^+(\psi^-(n)), \qquad n \in \mathbb{N},$$

so that each $f_i(1) = 1$. Since ϕ and ψ are strictly increasing in $\mathbb{U}$, each f_i is increasing in $\mathbb{N}$, although not necessarily strictly so. A natural question is: What mild conditions can be imposed on ϕ so that some f_i becomes the identity function on $\mathbb{N}$?

2. A condition on ϕ, and the sequences (h_n) and (k_n)

It turns out that if ϕ satisfies

$$\phi(n + 1) \geq \phi(n) + 1, \qquad n \in \mathbb{N}, \tag{1}$$

then f_1, f_2 are the identity function on $\mathbb{N}$, and f_3, f_4 are determined by two integer sequences (h_n) and (k_n), defined, for $n = 1, 2, \ldots$, by letting

$$\begin{aligned} h_n &= \text{the greatest } h \in \mathbb{N} \text{ such that } \psi(h) \leq n, \\ k_n &= \text{the least } k \in \mathbb{N} \text{ such that } \psi(k) \geq n. \end{aligned} \tag{2}$$

Both sequences (h_n) and (k_n) are strictly increasing: for suppose, if possible, that $h_n = h_{n+1}$ for some n, then the unit interval $n < x \leq n + 1$ would be free of images $\psi(h)$, so that there