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What is the Next Number in this Sequence?

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Given a sequence of numbers, or a sequence, in short, that is generated by an unknown polynomial or by a first-order recurrence relation, we describe an easy way to write down the polynomial or the recurrence relation by computing successive differences of the terms of the sequence.

What is the next number in each of the following sequences?

$$\begin{aligned} &S1: \ 1, 3, 8, 19, 42, 89, \ldots \\ &S2: \ 23, 48, 84, 133, \ldots \\ &S3: \ 7, 27, 58, 102, \ldots \end{aligned}$$

One may immediately protest that it is impossible to uniquely determine a sequence based only on its first few terms. R. K. Guy called this the 'strong law of small numbers' (see reference 1). Furthermore, it is well known that, given any n numbers, we can construct infinitely many polynomials $f(x)$ of degree n that evaluate to these n numbers when x takes on the value of the integers from 0 to $n-1$. As a result, C. E. Linderholm (see reference 2, p. 97) even suggested (with tongue firmly in cheek) that one should answer '19' to every 'find the next term' question.

Yet such questions continue to appear in IQ tests (the sequences S1, S2, and S3 are, respectively, puzzle 43 (from Test 1), puzzle 15 (from Test 6), and puzzle 17 (from Test 8), taken from reference 3, a book of IQ puzzles written by two UK Mensa puzzle editors). More importantly, they also appear in the school syllabus for the purpose of teaching number patterns and algebra. We must acknowledge, however, that such number pattern questions have been useful as training in recognising patterns and for the appreciation of mathematics. What is needed in such texts is an insertion somewhere that such questions as they stand are 'nonsense' if there is no assumption about the sequence. We should also add that a sequence is only defined once *all* the terms are given, either explicitly (which in this case is no fun because then there would be no problem) or in some other way (for example, stating that the numbers in the sequence are the consecutive digits of the decimal representation of π).

One way to avoid ambiguity in such number pattern problems is to restrict ourselves to only sequences generated by polynomials of degree k, and when posing the question, to give at least $k+1$ terms. With that assumption, the standard technique to make sense of a complicated number pattern is the method of calculating successive differences. Take the sequence S2.

S2	:	23		48		84		133
1st successive difference	:		25		36		49	
2nd successive difference	:			11		13		
3rd successive difference	:				2			

Here, the numbers in each new row are the difference of the two corresponding numbers of the previous row. Formally, if $a(n)$ denotes a sequence, then $a_i(n)$, the i th successive difference