

# On the Trail of Reverse Divisors: 1089 and All that Follow

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We determine all natural numbers that divide their reverses.

This is an account of the authors' adventures in tracking down, capturing, and cataloguing a rare species of whole number, known as a *reverse divisor*. So scarce are they that there are only 6 of them under a million, 16 under a billion, and 38 under a trillion. As the name suggests, a reverse divisor is a decimal integer that divides the number obtained by reversing its digits. It should be emphasized here that our whole discussion in the first two sections is based on the *decimal* expansion of a number. To avoid the commonplace, we exclude palindromic numbers, those that are unchanged by reversal of digits, in our formal definition: a *non-palindromic* natural number in decimal form  $ab \dots cd$  ( $a \neq 0$ ) which divides its *reverse*  $dc \dots ba$  is called a *reverse divisor*. For such a number, the quotient of the reverse by the original is called the *quotient of the reverse divisor*. Reverse divisors must have at least two digits, *cannot* end in 0, and the quotient of a reverse divisor is one of the numbers 2, 3, . . . , 9.

The first sighting of a reverse divisor is hard to come by. Single-digit numbers, being palindromic, do not qualify. A few minutes' mental arithmetic shows there are no two-digit reverse divisors, and even longer on a pocket calculator shows there are no three-digit reverse divisors. Rather than baldly announce the first reverse divisor, we invite the eagle-eyed amongst you to take stock and make an inspired stab in the dark at it, without lifting a finger. For those of you who were successful, and those who were not, please read on and join us on a mathematical journey, from knowing nothing about reverse divisors to knowing everything! We found it exciting, we hope that you will do so too.

The *only* reverse divisors having four or fewer digits are 1089 and its double 2178, with respective quotients 9 and 4, an observation that G. H. Hardy, the greatest English number theorist of the twentieth century, alludes to in his delightful *A Mathematician's Apology* (see pages 104–105 of reference 1) as *non-serious* mathematics. The *smallest reverse divisor* 1089 has over recent years become something of a *nombre célèbre* in recreational mathematics (see page 9 of reference 2 and page 163 of reference 3) on account of its following remarkable property, which you should try out for yourself, if you have not already done so.

*Subtract from any three digit number, whose first digit exceeds the last, its reverse. Add this difference to its own reverse. Then (in three-digit arithmetic) this last sum is always 1089.*

The result generalizes to four or more digit numbers (reference 4). A best-selling paperback by David Acheson (reference 5; see illustration overleaf) *actually* bears the title **1089 and All That**, despite not mentioning that 1089 is the *smallest reverse divisor*! Lewis Carroll entertained his child-friends with this arithmetical curiosity, and may even have been its discoverer (see pages 158–159 of reference 6). It appears under the heading **ABRACADABRA** in the News Chronicle's *I-SPY Annual* for 1956, whilst Johnny Ball in his fun maths book *Think of a Number* (see page 48 of reference 7) exploits it in a *mind-boggling* conjuring trick!