Poincaré and his Infamous Conjecture

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Disclaimer

Topology is a deep and beautiful subject. To provide precise statements and proofs assumes a certain degree of sophistication in the discipline. The current article makes no such assumption. Consequently, technicalities are omitted. Experts may shudder, but it is hoped that an audience with little background in topology will be grateful. For a more comprehensive account, the reader may consult references 1 or 2.

1. Topology

Elementary geometry was well established in ancient Greece by 300 BC, but there is a newer idea called topology. Imagine doing your homework not on paper, but on a sheet of stretchy rubber. Once you have finished, what would happen if you stretched the rubber? Concepts such as straight and parallel become meaningless – even size is challenged. Yet many interesting properties do make sense after stretching.

Let us illustrate this with an example. Picture a map of the London Underground. Does it tell you how far it is from King’s Cross to Euston, or the direction from one to the other? The answers are no, but millions of travellers navigate everyday with this topological map, which conveys how the stations are linked together. If you draw the Underground map on a rubber sheet, and then stretch the rubber, it will still tell you how the stations are configured.

In topology, we imagine that everything is made from rubber, and we allow ourselves to stretch and squash, but not to cut or glue. In terms of the rubber London Underground map, this means that you can distort the picture smoothly. Cutting the map might break links between stations, whilst gluing two Underground lines adds extra links. The confusion this would cause London’s population is why topologists forbid cutting and gluing!

If one rubber object can be manipulated to look just like another, we say that the two are topologically the same. For example, a rubber football could be squashed to look like a rugby ball, so they are topologically the same. Similarly, a handkerchief and a spinnaker sail are topologically the same. Can you suggest further pairs of objects which are topologically the same?

Note that when we talk about a football, we mean the skin or the surface of the ball, and not the air inside. Similarly, we distinguish between a circle and a solid circle or disc; see figure 1.

You may like to ask if the following phrase, common with French school children, is accurate from a topologist’s point of view. How about a geometer’s?

*Qu’est-ce qu’un cercle? Ce n’est point carré.*

(What is a circle? It’s not a square.)